OPTIMIZATION OF RENORMALIZATION GROUP FLOW

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ABSTRACT

Renormalization group flow equations for scalar $\lambda\Phi^4$ are generated using three classes of smooth smearing functions. Numerical results for the critical exponent ν in three dimensions are calculated by means of a truncated series expansion of the blocked potential. We demonstrate how the convergence of ν as a function of the order of truncation can be improved through a fine tuning of the smoothness of the smearing functions.

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I. INTRODUCTION

Renormalization group (RG) methods provide a powerful tool for investigating nonperturbative physical phenomena [1]. Issues such as QCD under extreme conditions, formation of a quark-gluon plasma in relativistic heavy-ion collisions [2], or critical phenomena in condensed matter systems all cannot be treated using standard perturbation theory due to the presence of infrared (IR) singularities. Through the continuous elimination of degrees of freedom, RG techniques systematically resum the perturbative series and therefore can provide information about nonperturbative effects in these systems. However, the power of RG relies on the existence of efficient analytic and computational methods since the full RG flow equations cannot be solved exactly. The goal of this paper is to show how to optimize the way the degrees of freedom are eliminated in order to improve results obtained using approximate RG flow equations.

For a field theoretical system a continuous RG transformation can be realized by introducing a smearing function $\rho_k(x)$ which governs the coarse-graining procedure [3] [4]. The scale k acts as an effective IR cutoff that separates the low- and high-momentum modes. Using this smearing function an averaged blocked field can be defined as

$$\phi_k(x) = \int_y \rho_k(x-y)\phi(y), \tag{1.1}$$

from which one obtains the effective Legendre blocked action:

$$e^{-\widetilde{S}_k[\Phi(x)]} = \int D[\phi] \prod_x \delta(\phi_k(x) - \Phi(\mathbf{x}))e^{-S[\phi]} \,. \tag{1.2}$$

In this manner we achieve a smooth interpolation between the bare action $S_\Lambda[\Phi]$ defined at the ultraviolet (UV) cutoff, Λ, and the quantum effective action $\widetilde{S}_k[\Phi]$, which generates those one-particle-irreducible graphs whose internal momenta extend between k and Λ [5]. The introduction of a smearing function leads to a modification of the bare propagator:

$$\Delta(p) = \frac{1}{p^2} \longrightarrow \Delta_k(p) = \frac{1-\rho_k(p)}{p^2} = \frac{\tilde{\rho}_k(p)}{p^2}, \tag{1.3}$$

with $\rho_k(p) + \tilde{\rho}_k(p) = 1$. One can then derive a RG equation for the blocked action which remains valid in all orders of the loop expansion by varying $\widetilde{S}_k$ infinitesimally with k:

$$\begin{aligned} k\frac{\partial \widetilde{S}_k}{\partial k} &= -\frac{1}{2}\mathrm{Tr}\left[\frac{1}{\Delta_k}\Big(k\frac{\partial \Delta_k}{\partial k}\Big)\Big(1 + \Delta_k\, \frac{\delta^2 \widetilde{S}_k}{\delta \Phi^2}\Big)^{-1}\right] \\ &= -\frac{1}{2}\mathrm{Tr}\left[\tilde{\rho}_k^{-1}\Big(k\frac{\partial \tilde{\rho}_k}{\partial k}\Big)\Big(1 + \frac{\tilde{\rho}_k}{p^2}\, \frac{\delta^2 \widetilde{S}_k}{\delta \Phi^2}\Big)^{-1}\right]. \end{aligned} \tag{1.4}$$

Since the pioneering work of Wilson [1], similar RG equations have been derived and analyzed. A sharp momentum cutoff was first used by Wegner and Houghton [6], and further discussed in Refs. [7], [8] and [9]. Polchinski [10], on the other hand, employed a smooth cutoff. There exists a vast amount of literature devoted to this subject [11]. Such functional RG equations, as stated before, have proven too difficult to be solved exactly, and further approximations are needed. Any viable scheme must not only retain the nonperturbative characteristics of the RG, but also converge sufficiently rapidly without inducing further spurious effects. As demonstrated by Morris in his seminal papers [5] [12], the most reliable method so far for probing the low-energy effective theory is the derivative expansion:

$$\widetilde{S}_k[\Phi] = \int_x \left\{ \frac{Z_k(\Phi)}{2}(\partial_\mu \Phi)^2 + U_k(\Phi) + O(\partial^4) \right\}, \tag{1.5}$$

where $Z_k(\Phi)$ and $U_k(\Phi)$ are, respectively, the wavefunction renormalization and the blocked potential. At leading order in the derivative expansion $Z_k(\Phi)$ is taken to be unity and the low-energy effective action is described solely in terms of $U_k(\Phi)$ [13]. This local potential approximation (LPA) results in the following flow equation:

$$k\frac{\partial U_k(\Phi)}{\partial k} = \frac{1}{2}\int_p \Big(k\frac{\partial \tilde{\rho}_k(p)}{\partial k} \Big) \frac{U_k''(\Phi)}{p^2 + \tilde{\rho}_k(p) U_k''(\Phi)}. \tag{1.6}$$

Although the above expression can be derived exactly [14], it may also be obtained by a differentiation of the perturbative one-loop result

$$\tilde{U}_k^{(1)}(\Phi) = \frac{1}{2}\int_p \ln \left\{ 1 + \tilde{\rho}_k(p) \frac{V''(\Phi)}{p^2} \right\} \tag{1.7}$$

with respect to $\ln(k)$ followed by a substitution $V''(\Phi) \to U_k''(\Phi)$ on the right-hand-side.

Clearly, the functional form of the RG equation for $U_k(\Phi)$ depends on the choice of the smearing function. A sharp cutoff, $\rho_k(p) = \Theta(k-p)$, provides a well-defined boundary between the high- and low-momentum modes and yields a nonlinear partial differential equation. On the other hand, for a general smooth cutoff, no clear separation exists and the RG flow remains an integro-differential equation. According to the universality principle the shape of the smearing function, $\rho_k(p)$, does not influence the physics of finite length scales. So long as the effective action contains all marginal or relevant operators the resulting RG flow must be scheme independent. At leading order in the derivative expansion, with no further approximation involved, physical results are indeed independent of the shape of $\rho_k(p)$, as demonstrated in [5] [15]. However, at next-to-leading order in the derivative expansion where the effect of $Z_k(\Phi)$ is included, the critical properties of the system will depend on whether $\rho_k(p)$ is sharp or smooth. In fact, ambiguities arise in the former, although self-consistency arguments can be used to circumvent this difficulty [16]. The dependence of η, the anomalous dimension, on the shape of the smearing function beyond leading order, has been calculated in [15] [17].

The critical exponents and other universal properties of the system may vary when they are determined in the vicinity of the fixed point of the truncated equation given in Eq. (1.6), or when the theory moves far away from the critical manifold. In fact, even if the solution includes all the relevant operators, the neglected irrelevant parameters are still evolving at the fixed point of the truncated equation. Since the modification of the irrelevant operators amounts to a change of an overall scale factor of the theory this scale factor remains cutoff dependent at the approximate fixed point. In order to minimize its variation one has to retain in the approximation the irrelevant operators with the critical exponent.

A frequently used technique is to expand $U_k(\Phi)$ in power series of Φ followed by a truncation at some order, thereby turning the problem into solving a set of coupled nonlinear ordinary differential equations. Although the numerical algorithms are simplified by this expansion, the convergence of critical exponents calculated in this way is not guaranteed. In fact, when employing a sharp smearing function while $U_k(\Phi)$ is expanded in Φ about the origin, the critical exponent ν has been shown to oscillate about its expected value as M, the number of terms in the series, is increased [12]. The convergence improves, however, if the expansion is made around the k-dependent minimum of $U_k(\Phi)$ [9]. Unfortunately, this approach fails when applied to an $O(N)$ field theory in the symmetry-broken phase due to the presence of the massless Goldstone modes which cause IR divergences to persist [14] [18]. It is therefore desirable to identify a computational method which does not suffer from these drawbacks. In addition, for gauge theories, the numerical complication of solving the non-truncated flow equations is even more overwhelming because of the proliferation of degrees of freedom and the complexity of their interactions. For these theories a polynomial truncation of the blocked potential seems inevitable. Thus, it is crucial to understand how to improve results obtained from truncation schemes.

In the present work, we examine what happens, in the truncated polynomial expansion of $U_k(\Phi)$, when a smooth smearing function $\rho_{k,\sigma}(p)$ with σ being the smoothness parameter, is utilized instead of $\Theta(k-p)$. In particular, we explore how ν changes with M as the smooth smearing functions approach the sharp limit, and how its convergence is influenced by σ. Our goal is to obtain a prescription which eliminates or diminishes the oscillations mentioned above and ensures a rapid convergence as M is increased.

The organization of the paper is as follows. In Sec. II we give three examples of smooth smearing functions and derive the corresponding RG equations. We show how the expected sharp cutoff limit can be recovered by recognizing that $\rho_k(p)$ is a *continuous* function in the vicinity $p \approx k$. Finite-temperature RG equations are also derived. In Sec. III we present the numerical solutions for ν associated with the three-dimensional Wilson-Fisher fixed point at various levels of polynomial truncation of $U_k(\Phi)$. In Sec. IV the source of scheme dependence on M and σ is discussed. We propose an optimized smooth smearing function which leads to a maximal cancellation of the effects of irrelevant operators and provides the fastest convergence of ν to a value which is in good agreement with the world's best estimate. Sec. V is reserved for summary and discussions.

II. SMOOTH CUTOFF FUNCTIONS

We begin by considering a general smooth representation of the smearing function $\rho_{k,\varepsilon}(p) = \Theta_\varepsilon(k,p)$ which approaches $\Theta(k-p)$ as $\varepsilon \to 0$. Due to the singular nature of the step function ambiguities can arise when taking the limit $\varepsilon \to 0$. More precisely, one will encounter terms involving $\Theta(0) \equiv \theta_0$. Since this value is not uniquely specified for the smooth parameterizations of the step function, it would seem that the approach to the sharp limit is not unique. The naive, or *mean*, approach in the limit $\varepsilon \to 0$ is:

$$\int_0^\infty dp\, \frac{\partial\Theta_\varepsilon(k,p)}{\partial k} G(\Theta_\varepsilon(k,p),p) \longrightarrow \int_0^\infty dp\, \delta(k-p) G(\theta_0,p) = G(\theta_0,k), \tag{2.1}$$

where $\partial_k\Theta_\varepsilon(k,p) \to \delta(k-p)$. This prescription, however, is oversimplified because the integrand of

$$\int_0^\infty dp\, G(\Theta_\varepsilon(k,p),p)), \tag{2.2}$$

is not uniformly convergent as $\varepsilon \to 0$. The proper treatment requires that $\Theta_\varepsilon(k,p)$ be used as an integration variable. Thus by setting $\Theta_\varepsilon(k,p) = \Theta_\varepsilon(1-p/k) = t$ and making a change of variables from p to t, we have

$$\int_0^\infty dp\, \frac{\partial\Theta_\varepsilon(k,p)}{\partial k} G(\Theta_\varepsilon(k,p),p) \longrightarrow \int_0^\infty dp\Big(-\frac{p}{k}\frac{dt}{dp}\Big) G(t,p) = -\int_{t(p=0)}^{t(p=\infty)} dt\, \frac{p(t)}{k}\, G(t,p(t)). \tag{2.3}$$

The sharp cutoff limit is obtained when $t(p=0)=1$, $t(p=\infty)=0$, $p(t)=k$ and Eq. (2.3) simplifies to

$$\int_0^\infty dp\, \frac{\partial\Theta_\varepsilon(k,p)}{\partial k} G(\Theta_\varepsilon(k,p),p) = \int_0^1 dt\, G(t,k). \tag{2.4}$$

In terms of t, the RG flow equation in Eq. (1.6) reduces to

$$\begin{aligned} k\frac{\partial U_k(\Phi)}{\partial k} &= \frac{S_d}{2}\int_{t(p=0)}^{t(p=\infty)} dt\, \frac{p^d(t)\, U_k''(\Phi)}{p^2(t)+(1-t)U_k''(\Phi)} \\ &= \frac{S_d k^d}{2}\int_{t(z=0)}^{t(z=\infty)} dt\, \frac{z^d(t)\, \bar{U}_k''(\Phi)}{z^2(t)+(1-t)\bar{U}_k''(\Phi)}, \end{aligned} \tag{2.5}$$

where $S_d = 2/(4\pi)^{d/2}\Gamma(d/2)$, $z(t) = p(t)/k$ and $\bar{U}_k''(\Phi) = U_k''(\Phi)/k^2$. Thus, with a given $\rho_{k,\varepsilon}(p) \equiv t$, we first invert the expression to obtain $z(t)$ and then make a substitution into Eq. (2.5) to deduce the corresponding smooth RG equation for $U_k(\Phi)$.

As stated in the Introduction, when a truncated polynomial expansion is used, physical quantities such as the critical exponents may depend on M as well as on σ. For a sharp cutoff, the critical exponent ν has been shown to exhibit oscillatory behavior about its expected value when $U_k(\Phi)$ is expanded about the origin or the k-dependent minimum [7][12], although the result improves significantly in the latter [9]. In order to determine whether these oscillations can be removed or reduced by using a smooth smearing function we consider below three parameterizations of $\rho_{k,\sigma}(p)$, all of which approach $\Theta(k-p)$, when the appropriate limit of the smoothness parameter σ is taken.

A. Hyperbolic Tangent

We first examine

$$\rho_{k,\varepsilon}(p) = \frac{1}{2}\Big[1 + \tanh\Big(\frac{k^2 - p^2}{pk\varepsilon}\Big)\Big]. \tag{2.6}$$

The above smearing function satisfies:

$$\begin{aligned}
&\text{(i)} \quad \lim_{\varepsilon\to 0} \rho_{k,\varepsilon}(p) = \Theta(k-p), \\
&\text{(ii)} \quad \lim_{k\to 0} \rho_{k,\varepsilon}(p) = \frac{1}{2}\big[1 + \tanh(-\infty)\big] = 0, \\
&\text{(iii)} \quad \lim_{k\to \infty} \rho_{k,\varepsilon}(p) = 1.
\end{aligned} \tag{2.7}$$

In this case the propagator is modified as:

$$\Delta_{k,\varepsilon}(p) = \frac{1}{p^2}\cdot\frac{1}{2}\Big[1 + \tanh\Big(\frac{p^2 - k^2}{pk\varepsilon}\Big)\Big]. \tag{2.8}$$

Substituting

$$k\frac{\partial \rho_{k,\varepsilon}}{\partial k} = \frac{k^2 + p^2}{2kp\varepsilon}\ \mathrm{sech}^2\Big(\frac{k^2 - p^2}{kp\varepsilon}\Big), \tag{2.9}$$

into Eq. (1.6) then yields:

$$k\frac{\partial U_k(\Phi)}{\partial k} = -\frac{S_d}{4\varepsilon}\int_0^\infty dp\ p^{d-1}\Big(\frac{k}{p} + \frac{p}{k}\Big)\ \mathrm{sech}^2\Big(\frac{k^2 - p^2}{kp\varepsilon}\Big)\frac{U_k''(\Phi)}{p^2 + \big[1 - \rho_{k,\varepsilon}(p)\big]U_k''(\Phi)}. \tag{2.10}$$

With the help of Eq. (2.4) we can rewrite the above expression as

$$k\frac{\partial U_k(\Phi)}{\partial k} = \frac{S_d k^d}{2}\int_{t(z=0)}^{t(z=\infty)} dt\ \frac{z^d(t)\ \bar{U}_k''(\Phi)}{z^2(t) + (1-t)\bar{U}_k''(\Phi)}, \tag{2.11}$$

where

$$\begin{aligned}
z(t) &= \sqrt{\frac{\varepsilon^2 a^2(t)}{4} + 1} - \frac{\varepsilon a(t)}{2} \\
a(t) &= \tanh^{-1}(2t-1).
\end{aligned} \tag{2.12}$$

One of our main goals here is to examine how the flow of the theory is influenced by tuning ε. As $\varepsilon \to 0$, the expected sharp-cutoff Wegner-Houghton equation [6]

$$k\frac{\partial U_k(\Phi)}{\partial k} = -\frac{S_d k^d}{2}\int_0^1 dt\ \frac{\bar{U}_k''(\Phi)}{1 + (1-t)\bar{U}_k''(\Phi)} = -\frac{S_d k^d}{2}\ln\Big[\frac{k^2 + U_k''(\Phi)}{k^2}\Big] \tag{2.13}$$

is recovered. As ε is increased the cutoff becomes smoother, i.e., the peak in $\partial\rho_{k,\varepsilon}(p)/\partial p$ at $p = k$ spreads. At about $\varepsilon \approx 2$ this tendency is reversed and the peak becomes taller again. The location of the peak stays roughly the same until $\varepsilon \approx 1$ and is shifted towards smaller values as ε is further increased. It finally moves to $p = k/\varepsilon$ for large ε. The decrease of the cutoff when $\varepsilon \to \infty$ eliminates the interactions and the theory flows into the trivial Gaussian fixed point. We shall comment more on this point later.

Notice that the mean approach described in Eq. (2.1) gives

$$
\begin{aligned}
k\frac{\partial U_k(\Phi)}{\partial k} &= \lim_{\varepsilon\to 0} -\frac{S_d}{4\varepsilon}\int_0^\infty dp\, p^{d-1}\Big(\frac{k}{p}+\frac{p}{k}\Big)\,\mathrm{sech}^2\Big(\frac{k^2-p^2}{kp\varepsilon}\Big)\frac{U_k''(\Phi)}{p^2+\big[1-\rho_{k,\varepsilon}(p)\big]U_k''(\Phi)} \\
&\longrightarrow -\frac{S_d}{2}\int_0^\infty dp\, p^{d-1}\; k\delta(k-p)\;\frac{U_k''(\Phi)}{p^2+\theta_0 U_k''(\Phi)} = -\frac{S_d k^d}{2}\;\Big(\frac{U_k''(\Phi)}{k^2+\theta_0 U_k''(\Phi)}\Big),
\end{aligned}
\tag{2.14}
$$

which differs from the Wegner-Houghton equation in its absence of the characteristic logarithmic functonal structure. In fact, the RG equation is highly sensitive to the k dependence in $\rho_{k,\sigma}(p)$. Nevertheless, we shall see that the difference between the two equations does not affect the critical properties significantly in the next section.

Next, we turn the nonlinear flow of $U_k(\Phi)$ into a set of coupled ordinary differential equations by making an expansion in power series of Φ:

$$
U_k(\Phi) = \sum_{\ell=1}^{\infty}\frac{g_k^{(2\ell)}}{(2\ell)!}\;\Phi^{2\ell}, \qquad\qquad g_k^{(2\ell)} = U_k^{(2\ell)}(0) = \frac{\partial^{2\ell}U_k}{\partial\Phi^{2\ell}}\Big|_{\Phi=0}, \tag{2.15}
$$

followed by a truncation at some order $\ell = M$. The resulting running equations for the first two terms read:

$$
\begin{aligned}
k\frac{\partial\bar\mu_k^2}{\partial k} &= -2\bar\mu_k^2 + J_0\bar\lambda_k, \\
k\frac{\partial\bar\lambda_k}{\partial k} &= -\epsilon\bar\lambda_k - \frac{6\bar\lambda_k^2}{\bar\mu_k^2}\big(J_0 - J_1\big) + J_0\bar g_k^{(6)},
\end{aligned}
\tag{2.16}
$$

where $\epsilon = 4 - d$,

$$
J_n(\bar\mu_k^2) = -\frac{S_d}{2}\int_0^1 dt\;\frac{z(t)^{d+2(n+1)}}{\big[z^2(t)+(1-t)\bar\mu_k^2\big]^{2+n}}, \tag{2.17}
$$

and $z(t)$ is given by Eq. (2.12). In the sharp cutoff limit where $z(t)\to 1$ and

$$
J_n(\bar\mu_k^2) = \frac{S_d}{2(1+n)}\;\frac{1-(1+\bar\mu_k^2)^{1+n}}{\bar\mu_k^2(1+\bar\mu_k^2)^{1+n}}, \tag{2.18}
$$

we recover

$$
\begin{aligned}
k\frac{\partial\bar\mu_k^2}{\partial k} &= -2\bar\mu_k^2 - \frac{S_d}{2}\frac{\bar\lambda_k}{1+\bar\mu_k^2}, \\
k\frac{\partial\bar\lambda_k}{\partial k} &= -\epsilon\bar\lambda_k + \frac{S_d}{2}\Big\{\frac{3\bar\lambda_k^2}{\big(1+\bar\mu_k^2\big)^2} - \frac{\bar g_k^{(6)}}{1+\bar\mu_k^2}\Big\}.
\end{aligned}
\tag{2.19}
$$

This can be compared with the result obtained using the mean approach:

$$
\begin{aligned}
k\frac{\partial \bar{\mu}_k^2}{\partial k} &= -2\bar{\mu}_k^2 - \frac{S_d}{2}\frac{\bar{\lambda}_k}{\left(1+\theta_0\bar{\mu}_k^2\right)^2},\\
k\frac{\partial \bar{\lambda}_k}{\partial k} &= -\epsilon\bar{\lambda}_k + \frac{S_d}{2}\left\{\frac{6\theta_0\bar{\lambda}_k^2}{\left(1+\theta_0\bar{\mu}_k^2\right)^3} - \frac{\bar{g}_k^{(6)}}{\left(1+\theta_0\bar{\mu}_k^2\right)^2}\right\}.
\end{aligned}
\tag{2.20}
$$

A comparison of the two flow equations shows that in order to recover the standard perturbative RG coefficient functions in the UV limit $\bar{\mu}_k^2 \ll 1$, the usual convention $\theta_0 = 1/2$ must be imposed. However, this choice also implies a deviation of scaling for a massive theory in the IR regime where $\bar{\mu}_k^2 \gg 1$.

It is easy to understand that the two flows agree in the UV scaling regime only. In fact, the non-uniform convergence in Eq. (2.2) leads to an "incorrect" use of the integrand of the evolution equation Eq. (1.6) at $p \approx k$ when the limit $\varepsilon \to 0$ is taken before the integration. Since the integrand is independent of the field Φ for $k^2 \gg U_k''(\Phi)$ the mistake is unimportant. But the naive evolution equation displays wrong $U_k''(\Phi)$ dependence when $k^2 \approx U_k''(\Phi)$.

B. Exponential Function

An alternative class of smearing functions is the exponential function

$$
\rho_{k,b}(p) = e^{-a(p/k)^b}, \tag{2.21}
$$

where a and b are constants. This smearing function satisfies the following conditions:

$$
\begin{aligned}
&\text{(i)} \quad \lim_{b\to\infty} \rho_{k,b}(p) = \Theta(k-p),\\
&\text{(ii)} \quad \lim_{k\to 0} \rho_{k,b}(p) = 0,\\
&\text{(iii)} \quad \lim_{k\to\infty} \rho_{k,b}(p) = 1.
\end{aligned}
\tag{2.22}
$$

In order to facilitate comparison with the other smearing functions used here, we also require $\rho_{k,b}(p=k) = 1/2$, which in turn implies $a = \ln 2$, or $\rho_{k,b}(p) = 2^{-(p/k)^b}$.

The corresponding RG equation reads

$$
k\frac{\partial U_k(\Phi)}{\partial k} = -\frac{S_d k^d}{2}\, ab \int_0^\infty dy\; y^{d+b-1}\, \frac{e^{-ay^b}\bar{U}_k''(\Phi)}{y^2 + \left(1-e^{-ay^b}\right)\bar{U}_k''(\Phi)}. \tag{2.23}
$$

Alternatively, one can also cast Eq. (2.23) into the same form as Eq. (2.11), but with $z(t)$ given by

$$
z(t) = \left(-\frac{\ln t}{a}\right)^{1/b}. \tag{2.24}
$$

In the asymptotic limit where $\bar{U}_k''(\Phi) \ll 1$, the RG flow can be approximated as

$$k\frac{\partial U_k}{\partial k} \approx -\frac{S_d k^d}{2}\Big[c_1\ \frac{U_k''}{k^2} + c_2 \frac{U_k''^2}{k^4} + O(\frac{U_k''^3}{k^6})\Big], \tag{2.25}$$

where

$$\begin{aligned} c_1 &= a^{-(d-2)/b}\ \Gamma\big(\frac{b+d-2}{b}\big), \\ c_2 &= -\big[1-2^{-(d+b-4)/b}\big]a^{-(d-4)/b}\ \Gamma\big(\frac{b+d-4}{b}\big). \end{aligned} \tag{2.26}$$

The sharp limit, on the other hand, gives $c_1 = 1$ and $c_2 = -1/2$ independent of dimensionality d. Thus, c_1 and c_2 provide a measure of the deviation from the sharp-cutoff UV scaling. We also note that in order to avoid UV singularities in the integrations, b must be chosen as to avoid singularities in the Gamma functions above. As b is decreased the cutoff becomes smoother. The location of the peak in $\partial\rho_{k,b}(p)/\partial p$ is stable for large b and starts to shift at $b \approx 3$. For $b \lesssim 3$, the shape of $\rho_{k,b}(p)$ spreads in such a manner that one can no longer associate a well-defined cutoff value.

C. Power-law Function

Lastly, we consider a power-law smearing function:

$$\rho_{k,m}(p) = \frac{1}{1+(p/k)^m}, \tag{2.27}$$

which satisfies

$$\begin{aligned} &\text{(i)} \quad \lim_{m\to\infty} \rho_{k,m}(p) = \Theta(k-p), \quad [\ \theta_0 = 1/2 \quad \text{when } p = k\ ], \\ &\text{(ii)} \quad \lim_{k\to 0} \rho_{k,m}(p) = 0, \\ &\text{(iii)} \quad \lim_{k\to\infty} \rho_{k,m}(p) = 1. \end{aligned} \tag{2.28}$$

The RG equation in this case reads

$$\begin{aligned} k\frac{\partial U_k(\Phi)}{\partial k} &= -\frac{S_d k^d}{2}\ m\int_0^\infty dy\ y^{d+m-1}\ \frac{\bar{U}_k''(\Phi)}{\big(1+y^m\big)\big[y^2(1+y^m)+y^m\ \bar{U}_k''(\Phi)\big]} \\ &= -\frac{S_d k^d}{2}\int_0^1 dt\ \frac{z^d(t)\ \bar{U}_k''(\Phi)}{z^2(t)+(1-t)\bar{U}_k''(\Phi)}, \end{aligned} \tag{2.29}$$

where $y = p/k$ and

$$z(t) = \Big(\frac{1}{t}-1\Big)^{1/m}. \tag{2.30}$$

The sharp limit, i.e., $z \to 1$, may also be obtained by the substitution:

$$\frac{my^{m-1}}{\big(1+y^m\big)^2}\ G\big(\frac{1}{1+y^m}, p\big) \longrightarrow \delta(1-y)\int_0^1 dt\ G(t,p), \quad m\to\infty, \tag{2.31}$$

which results from

$$\lim_{m\to\infty} \frac{my^{m-1}}{(1+y^m)^2} = \delta(1-y). \tag{2.32}$$

The corresponding flow equations for $\bar{\mu}_k^2$ and $\bar{\lambda}_k$ have the same functional forms as those in Eq. (2.20) but with

$$J_n(\bar{\mu}_k^2) = -\frac{S_d}{2}\, m \int_0^\infty dz\ \frac{(1+z^m)^n z^{d+2n+m+1}}{\left[z^2(1+z^m) + z^m \bar{\mu}_k^2\right]^{2+n}}. \tag{2.33}$$

On the other hand, if one adopts the mean approach and takes the limit $m \to \infty$, the RG equation then becomes

$$k\frac{\partial U_k(\Phi)}{\partial k} \to -\frac{S_d k^d}{2} \int_0^\infty dy\ \delta(1-y) \frac{y^d U_k''(\Phi)}{y^2 + \Theta(y-1)U_k''(\Phi)} = -\frac{S_d k^d}{2} \frac{U_k''(\Phi)}{k^2 + \theta_0 U_k''(\Phi)}, \tag{2.34}$$

which again coincides with Eq. (2.14). The peak in $\partial\rho_{k,m}(p)/\partial p$ is stable at $p = k$ for $m > 5$ but shifts towards lower momenta for $m < 5$, and the shape of the smearing function makes it difficult to identify a clear cutoff value.

Unlike the previous two parameterizations where the integro-differential flow equations must be solved numerically, for certain values of d and m analytic evaluation of the integral on the right-hand-side of Eq. (2.29) is possible. For example, taking $d = m = 4$ we have

$$k\frac{\partial U_k}{\partial k} = -\frac{k^4}{16\pi^2} \frac{U_k''}{\sqrt{4k^4 - U_k''^2}} \left\{\pi - 2\tan^{-1}\left(\frac{U_k''}{\sqrt{4k^4 - U_k''^2}}\right)\right\}. \tag{2.35}$$

And for $d = 3$, we have

$$k\frac{\partial U_k(\Phi)}{\partial k} = \begin{cases} -\dfrac{k^3}{4\pi}\left[-1 + \sqrt{1 + U_k''(\Phi)/k^2}\,\right], & (m=2), \\[2ex] -\dfrac{k^3}{4\pi} \dfrac{1}{\sqrt{2 + U_k''(\Phi)/k^2}}\left[-2 + \sqrt{4 + 2U_k''(\Phi)/k^2}\,\right], & (m=4). \end{cases} \tag{2.36}$$

In this differential form it is possible to apply the standard techniques of differential equations to solve for the RG flow of the theory.

In the asymptotic limit where $\bar{U}_k''(\Phi) \ll 1$, Eq. (2.29) becomes

$$k\frac{\partial U_k}{\partial k} \approx -\frac{S_d k^d}{2}\left[\tilde{c}_1\ \frac{U_k''}{k^2} + \tilde{c}_2 \frac{U_k''^2}{k^4} + O\left(\frac{U''^3_k}{k^6}\right)\right], \tag{2.37}$$

where

$$\begin{aligned} \tilde{c}_1 &= \frac{m}{d+m-2}\Gamma\left(\frac{m-d+2}{m}\right)\Gamma\left(\frac{2m+d-2}{m}\right) \overset{m\to\infty}{\longrightarrow} 1, \\ \tilde{c}_2 &= -\frac{m}{2(d+2m-4)}\Gamma\left(\frac{m-d+4}{m}\right)\Gamma\left(\frac{3m-d+4}{m}\right) \overset{m\to\infty}{\longrightarrow} -\frac{1}{2}. \end{aligned} \tag{2.38}$$

Before closing this section, we remark that the RG techniques discussed so far can be readily extended to finite-temperature systems. For the scalar $\lambda\Phi^4$ theory defined on $S^1 \times R^d$ in the imaginary-time formalism, with the radius of S^1 being given by β, the inverse temperature, one obtains

$$
\begin{aligned}
k\frac{\partial U_{\beta,k}(\Phi)}{\partial k} &= -\frac{S_d k^d}{2\beta}\left\{\beta\sqrt{k^2+U''_{\beta,k}(\Phi)} + 2\,\ln\Big[1-e^{-\beta\sqrt{k^2+U''_{\beta,k}(\Phi)}}\Big]\right\} \\
&= -\frac{S_d k^d}{\beta}\,\ln\sinh\left(\frac{\beta\sqrt{k^2+U''_{\beta,k}(\Phi)}}{2}\right)
\end{aligned}
\qquad (2.39)
$$

using a sharp cutoff [16]. A smooth smearing function, on the other hand, yields an integro-differential equation:

$$
\begin{aligned}
k\frac{\partial U_{\beta,k}(\Phi)}{\partial k} &= -\frac{1}{4}\int_{\mathbf{p}}\left(k\frac{\partial \rho_{k,\varepsilon}}{\partial k}\right)\frac{U''_{\beta,k}}{\sqrt{\mathbf{p}^2+\big(1-\rho_{k,\varepsilon}\big)U''_{\beta,k}}}\,\coth\left(\frac{\beta\sqrt{\mathbf{p}^2+(1-\rho_{k,\varepsilon})U''_{\beta,k}}}{2}\right) \\
&\longrightarrow \frac{S_d k^d}{4}\int_{t(z=0)}^{t(z=\infty)} dt\;\frac{z^d(t)\bar{U}''_{\beta,k}}{\sqrt{z^2(t)+(1-t)\bar{U}''_{\beta,k}}}\,\coth\left(\frac{\bar{\beta}\sqrt{z^2(t)+(1-t)\bar{U}''_{\beta,k}}}{2}\right),
\end{aligned}
\qquad (2.40)
$$

where $\bar{\beta} = \beta k$. The two equations coincide in the sharp cutoff limit. In the high-temperature regime where $\bar{\beta} \to 0$, we have

$$
k\frac{\partial U_{\beta,k}(\Phi)}{\partial k} \approx \frac{S_d k^d}{2}\int_{t(z=0)}^{t(z=\infty)} dt\; z^d(t)\left[\frac{\bar{U}''_{\beta,k}(\Phi)}{\bar{\beta}\big[z^2(t)+(1-t)\bar{U}''_k(\Phi)\big]} + \frac{\bar{\beta}}{12}\bar{U}''_{\beta,k}(\Phi) + O(\bar{\beta}^3)\right]. \qquad (2.41)
$$

Apart from the factor of $\bar{\beta}^{-1}$, the first term in the bracket corresponds to the zero-temperature d-dimensional case, in accord with the expectation of dimensional reduction. In this limit, $\bar{\beta}$ can be scaled away with a redefinition of $U_{\beta,k}(\Phi) \to \beta^{-1}U_k(\Phi)$ and $\Phi \to \beta^{-1/2}\Phi$ [16] [19].

III. FIXED-POINT SOLUTIONS AND NUMERICAL RESULTS

As seen in the last section, when the smearing function is smooth the flow of the theory is generally characterized by an integro-differential equation, as opposed to the differential equation obtained with the sharp cutoff or special cases of a smooth cutoff. In dimensionless form, the flow equation reads:

$$
\left[k\frac{\partial}{\partial k} - \frac{1}{2}(d-2)\bar{\Phi}\frac{\partial}{\partial\bar{\Phi}} + d\right]\bar{U}_k(\bar{\Phi}) = -\int_0^1 dt\;\frac{z^d(t)\;\bar{U}''_k(\bar{\Phi})}{z^2(t)+(1-t)\bar{U}''_k(\bar{\Phi})}, \qquad (3.1)
$$

where $\bar{U}_k(\bar{\Phi}) = \zeta^2 k^{-d} U_k(\Phi)$, $\bar{\Phi} = \zeta k^{-(d-2)/2}\Phi$, $\zeta = \sqrt{2/S_d} = \sqrt{(4\pi)^{d/2}\Gamma(d/2)}$, and the momentum scale $z(t)$ is given by Eqs. (2.12), (2.24) or (2.30) for the three cases considered. That S_d can be absorbed by a redefinition of $\bar{\Phi}$ is an indication of universality, i.e., critical exponents are independent of S_d, though the matrix elements in the linearized RG matrix are. In the above, the anomalous dimension η has been set to zero since we are only considering the flow of the potential.

To characterize the critical behavior of the theory, our first task is to identify all the fixed points and then linearize RG about a particular fixed point. It is well known that for the one-component scalar theory in $d = 3$, there are only two fixed points: one trivial Gaussian and one Wilson-Fisher, and the critical behavior is dominated by the latter. No other continuum limit is known to exist. This applies to both the sharp and the smooth cutoffs. However, for a general smooth function $\rho_{k,\sigma}(p)$, the location of the Wilson-Fisher fixed point which will now depend on σ; at $\sigma = 0$, however, it coincides with the Gaussian one.

At the fixed point(s), the theory exhibits scale invariance, i.e. $\partial_k \bar{U}_k^* = 0$, and the RG equation for the fixed point potential becomes (dropping the subscript k):

$$
\begin{aligned}
-\frac{1}{2}\big(d-2\big)\bar{\Phi}\bar{U}^{*'}(\bar{\Phi}) + d\bar{U}^*(\bar{\Phi}) &= -\int_0^1 dt \, \frac{z^d(t)\ \bar{U}^{*''}(\bar{\Phi})}{z^2(t) + (1-t)\bar{U}^{*''}(\bar{\Phi})} \\
&\overset{\text{s.}}{\longrightarrow} \ -\ln\Big[1 + \bar{U}^{*''}(\bar{\Phi})\Big],
\end{aligned}
\tag{3.2}
$$

where the notation s. stands for the sharp cutoff limit. Analytically, the sharp limit has the following approximate non-trivial solution in the large $\bar{\Phi}$ limit for $d = 3$:

$$
\bar{U}^*(\bar{\Phi}) = A\bar{\Phi}^6 - \frac{4}{3}\ln\bar{\Phi} - \frac{2}{9} - \frac{1}{3}\ln(30A) - \frac{1}{150A\bar{\Phi}^4} + O(\bar{\Phi}^{-6}),
\tag{3.3}
$$

where A is an arbitrary positive constant [12]. The RG flow about Eq. (3.3) can be linearized by writing $\bar{U}_k(\bar{\Phi}) = \bar{U}^*(\bar{\Phi}) + \bar{v}(\bar{\Phi})e^{-\lambda \ln k}$, where $\bar{v}(\bar{\Phi})$ obeys

$$
\begin{aligned}
-\frac{1}{2}(d-2)\bar{\Phi}\bar{v}'(\bar{\Phi}) - \big(\lambda - d\big)\bar{v}(\bar{\Phi}) &= -\int_0^1 dt\, \frac{z^d(t)\bar{v}''(\bar{\Phi})}{\big[z^2(t) + (1-t)\bar{U}^{*''}(\bar{\Phi})\big]^2} \\
&\overset{\text{s.}}{\longrightarrow} \ -\frac{\bar{v}''(\bar{\Phi})}{1 + \bar{U}^{*''}(\bar{\Phi})}.
\end{aligned}
\tag{3.4}
$$

We comment that the location of the fixed point as well as $\bar{U}^*(\bar{\Phi})$, being non-universal, naturally all depend on the choice of the smearing function $\rho_k(p)$. In the one extreme where $\rho_{k,\sigma}(p) = \Theta(k-p)$, linearlizing RG and retaining only the relevant operators yields

$$
(\bar{\mu}^{2*}, \bar{\lambda}^*) = (0,0), \quad \Big(-\frac{\epsilon}{6+\epsilon}, \frac{12\epsilon}{B(6+\epsilon)^2}\Big),
\tag{3.5}
$$

where $B = 1/16\pi^2$. However, in the other extreme where the smearing function is taken to be a k-independent constant, $\rho_{k,\sigma}(p) = c$, the right-hand-side of Eq. (3.1) then vanishes, and the RG flow simplifies to

$$
\frac{dU_k(\Phi)}{dk} = 0 \quad \Longrightarrow \quad -\frac{1}{2}\big(d-2\big)\bar{\Phi}\bar{U}^{*'}(\bar{\Phi}) + d\bar{U}^*(\bar{\Phi}) = 0.
\tag{3.6}
$$

The equation can be exactly solved to give $\bar{U}^*(\bar{\Phi}) = A\bar{\Phi}^{2d/(d-2)}$. For $\epsilon = 1$ or $d = 3$, $U^* \sim \bar{\Phi}^6$. Thus, with vanishing contributions from quantum fluctuations, the only fixed point is Gaussian $(\bar{\mu}^{2*}, \bar{\lambda}^*) = (0, 0)$. However, this does not mean that the theory is free, because we still have $\bar{U}^*(\bar{\Phi}) \neq 0$.

For the general case of a smooth cutoff function, however, it is rather difficult to obtain a non-truncated solution for Eq. (3.2), and we circumvent the problem with polynomial truncation at order ϕ^{2M}. The remaining task is to solve a system of M integro-differential equations for the coupling constants $g^{(2\ell)*}$, $\ell = 1, \cdots, M$. We first determine the fixed-point solution analytically in the sharp cutoff limit, so that we can then use this as an initial guess in our numerical root-finding subroutine for the smooth case. The system of integro-differential equations is solved by first choosing a large value for the smoothness parameter σ ($1/\varepsilon$, b, or m), thereby making the smearing function rather sharp. The smoothness of the cutoff function is increased in small steps by decreasing the parameters, and the solution of the previous step is used as the guess for finding the solution at the current step. In this manner we are able to track the solution associated with a physical fixed point from the sharp cutoff limit to an arbitrary smoothness.

With the strategies outlined above, we now present the results for all three smearing functions considered in Sec. II.

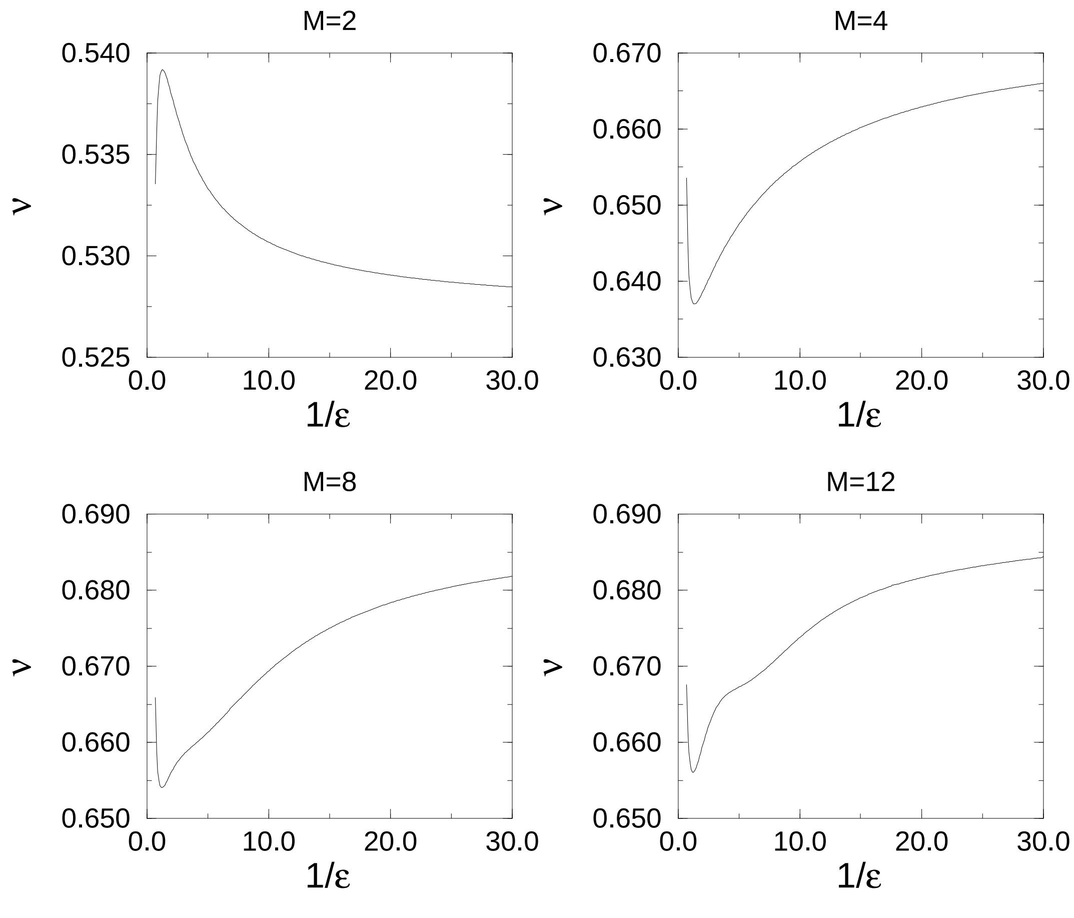

Figure 1. Critical exponent ν as a function of $1/\varepsilon$ using the hyperbolic tangent smearing function. Results for four different levels of polynomial truncation are shown.

In Fig. 1 we plot the dependence of ν in $d = 3$ as a function of the (inverse) smoothness parameter ε^{-1} at four different levels of truncation of $U^*(\Phi)$. From the figure, we see that ν varies by 2-5% over the range shown. The variation would become even larger had we taken $\varepsilon^{-1} \to 0$, where the smearing function becomes very flat and there is practically no blocking. The exponents in this limit are those obtained in the mean-field approximation, e.g., $\nu = 0.5$. It is also interesting to note that at each order in M there is a dip, or an extremum whose position changes only slightly with M. The non-monotonic behavior shows that when going to the sharp limit by gradually increasing ε^{-1}, ν first deviates from the sharp result and then begins to converge when $\rho_{k,\varepsilon}(p)$ becomes sufficiently sharp.

When an exponential smearing function is employed instead, we observe a similar behavior for ν, as depicted in Figure 2. Once more, the value varies by 2-5% and an extremum is found for each M. The qualitative feature remains the same for a power-law smearing function, as demonstrated in Figure 3. However, in this case the variation of ν is only about 2-3%.

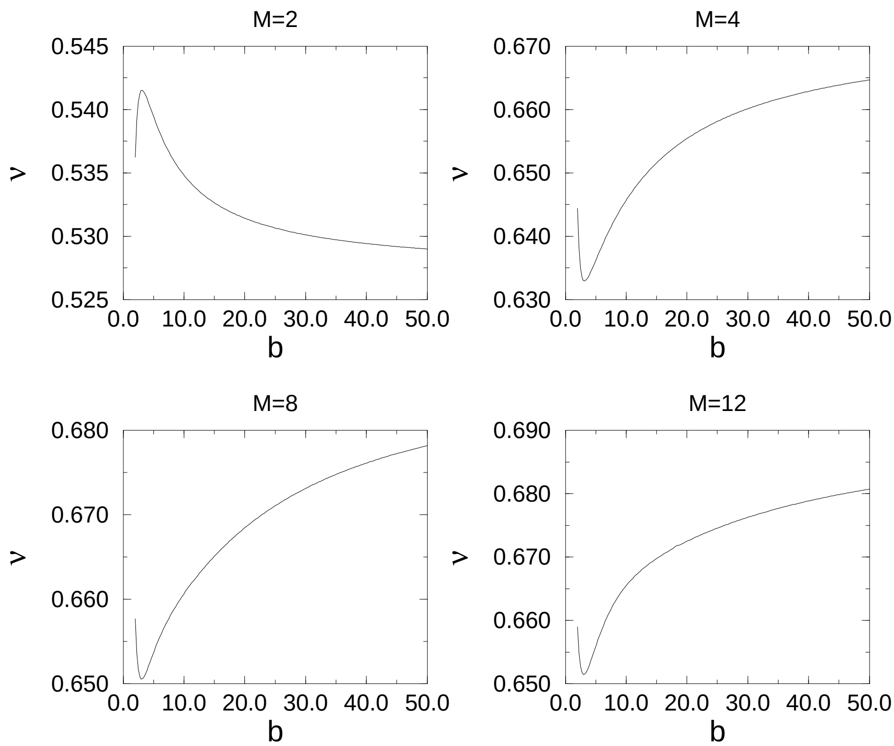

Figure 2. Critical exponent ν as a function of b for the exponential smearing function. Results for four different levels of polynomial truncation are shown.

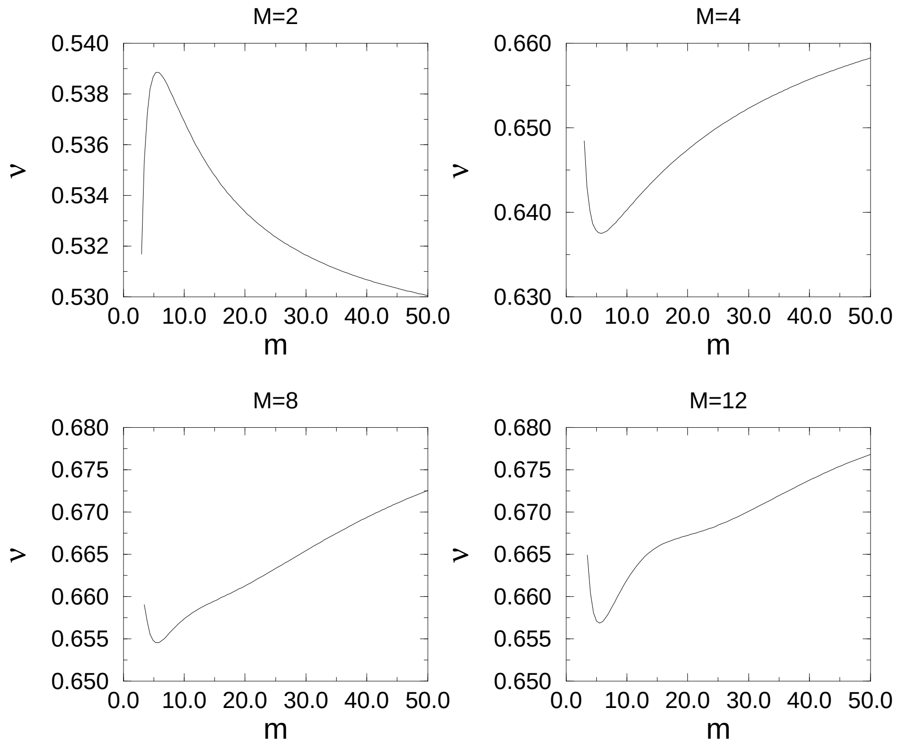

Figure 3. Critical exponent ν as a function of m for the power law smearing function. Results for four different levels of polynomial truncation are shown.

IV. SCHEME DEPENDENCE

Even though non-truncated solutions for the fixed-point potential $\bar{U}^*(\bar{\Phi})$ can be obtained in special cases [12], this is not possible in general. Any approximation inevitably brings in scheme-dependent effects. Our results are seen to depend both on the order of truncation M as well as the smoothness parameter σ. We discuss below the implications of such dependence.

A. Truncation Dependence

At order M we obtain M fixed-point solutions parameterized by the critical coupling constants $(g^{(2)*} \cdots g^{(2M)*})$, each having its own eigenvectors and eigenvalues which in turn can be used for calculating the critical exponents. However, only one of the solutions corresponds to the Wilson-Fisher fixed point which is characterized by one relevant operator, with the rest being irrelevant. The trivial Gaussian solution with all $g^{(2\ell)*} = 0$ is also obtained but is of no interest to us. Computational artifacts are induced with a polynomial truncation of the blocked potential $\bar{U}_k(\bar{\Phi})$ [9][12][16]. For example, as M is increased, numerous unphysical fixed points are also generated.

Thus, to isolate the physically meaningful solution from the unphysical ones in the smooth parameterization, what we have done numerically was to first analyze the sharp cutoff limit of Eq. (3.2) and identify the solution associated with the Wilson-Fisher fixed point. The salient feature of this solution is that it is stable against an increase in M. In other words, the value of $g^{(2\ell)*}$ is not sensitive to whether higher order coupling constants, e.g., $g^{(2\ell+2)*}$, are included or not. On the other hand, unphysical solutions are sensitive to the higher order equations and their eigenvalues will fluctuate in an unpredictable manner [8].

Another prominent artifact of the polynomial truncation is the oscillation of the critical exponents with M when a sharp cutoff is used. The values of the critical coupling constants also alternate in sign. The oscillations are due to the nonanalyticity of the critical blocked potential $\bar{U}^*(\bar{\Phi})$ and can be understood from the fact that an expansion of the right-hand-side of the sharp cutoff flow equation $\ln(1 + \bar{U}^{*''}(\bar{\Phi})) = \sum_{\ell=1}^{\infty}(-1)^{(\ell+1)}(\bar{U}^{*''}(\bar{\Phi}))^{\ell}/\ell$ also yields a power series of $\bar{\Phi}$ with alternating sign. Even though the convergence has been shown to improve significantly by expanding $\bar{U}_k(\bar{\Phi})$ around its moving minimum, there still remains residual scheme dependence [9]. In addition, an expansion about the minimum fails in the broken phase of $O(N)$ as stated in the Introduction. Therefore, it is desirable to have a numerical technique which allows for an efficient determination of the critical properties, and remove or reduce the aforementioned spurious effects.

Another subtle issue remains in this approximation is: why can we not reproduce the exact critical exponent(s) by keeping only the renormalizable coupling constants in our computation and in the identification of the fixed point? The main reason lies in that fact the fixed point of the truncated solution is applicable only up to the operators which have been neglected. At the point which we call a "fixed point," the would-be ignored set of irrelevant operators does not vanish but continues to evolve. On the other hand, in the exact RG approach, the fixed-point condition implies a complicated cancellation between different operators. When all the operators are present, the cancellation is complete and we should be able to eliminate the $\rho_{k,\sigma}(p)$-dependence in the scaling laws around the fixed point as well as in the expression for the critical exponents.

B. Smoothness Dependence

In the alternative Wilson or Polchinski RG approach, one can show that when the effective action is expanded in terms of derivatives to order n, only $2n$ parameters are required to absorb the scheme dependence [15]. Therefore, at the leading order of $U_k(\Phi)$ with $n = 0$, the critical exponents must be scheme-independent, i.e., the same results are obtained regardless of whether the smearing function is sharp, exponential or power-like. In fact, $\rho_{k,\sigma}(p)$ can be conveniently absorbed by a suitable redefinition of the field variable [20]. Scheme depedence appears, however, when the wavefunction renormalization $Z_k(\Phi)$ is taken into consideration at the next order.

On the other hand, Eq. (3.1) derived with LPA does not possess this scaling property, and the functional form of the flow equation is explicitly dependent of $\rho_{k,\sigma}(p)$. The dependence, nevertheless, is expected to be negligible in the calculation of physical quantities such as the critical exponents. This universality hypothesis, however, must be substantiated by solving the integro-differential equation Eq. (3.1) exactly without polynomial expansion.

From Figures 1, 2 and 3, the critical exponent ν clearly depends on the smoothness parameter σ (ε^{-1}, b and m). This comes as a consequence of polynomial truncation. That is, because of the polynomial truncation scheme employed, physical results now depend both on the level of truncation M as well as on σ.

How can such dependence be reconciled with universality? The key to the problem here again lies in the role played by the irrelevant operators in the approximate solution. If we solve the RG equation for the full theory and come down to the IR regime, the dependence on the initial condition for the irrelevant operators must die out according to the universality hypothesis. The explicit verification of this scenario by reaching the IR fixed point in principle requires an infinite number of iterations. This is not what was followed in this work. Instead we move close to the UV fixed point and a linearized RG prescription is utilized to deduce the critical exponents and the corresponding scaling laws. Using the derivative expansion and the polynomial truncation schemes gives the advantage of a readily accessible (approximate) fixed-point solution; however, we are no longer certain what the omitted irrelevant operators do. If they are not scale invariant at the fixed point - the likely result of the truncation, then their variation amounts to the change of the overall scale factor of the exact solution of the theory. This is because the irrelevant coupling constant set of the theory influences an overall scale factor. In order to minimize this scale dependence the truncation scheme must be "improved".

The comparison with the exact RG offers a direction of improvement in our approximation: to minimize the number and the strength of the neglected irrelevant operators. As far as the $\rho_{k,\sigma}(p)$-dependence is concerned, the momentum dependent, i.e. higher order derivative terms come into consideration which are quadratic in the field Φ. Therefore, we aim to minimize the number of such terms generated in the RG flow. The cutoff-function $\rho_{k,\varepsilon}(p)$ is strongly momentum dependent in an interval $\Delta p = \varepsilon k$. According to the uncertanity principle this yields non-local interactions on the length scale $\Delta x = 1/\varepsilon k$. In order to eliminate such a higher order derivative effect from the theory we must look for a cutoff which is as smooth as possible in the momentum space. This would yield a better convergence when the irrelevant derivative operators are truncated, in accord with the spirit of LPA.

However, there are also higer-order field operators and we should keep M as low as possible. We emphasize that the purpose of employing a truncated polynomial expansion of $U_k(\Phi)$ is to devise a simple algorithm which allows us to reproduce the nonperturbative features, particularly the fixed-point structure embedded in LPA, or even the blocked action $\widetilde{S}_k[\Phi]$, without having to solve the full flow equation for $U_k(\Phi)$. Thus, to go to a large M not only is computationally laborious but also in conflict with the original goal.

In Fig. 4 the truncation dependence of ν obtained using the exponential smearing function given in Eq. (2.21) is depicted. The sharp cutoff results are also included for comparative purpose. We notice the general trend of a more pronounced oscillatory behavior as b is increased toward the sharp limit. This again show that the sharp cutoff is not the *best* candidate if we look for the smallest set of operators which yields the most rapid convergence for the physical observables. Below we demonstrate how this can be achieved with an optimization scheme.

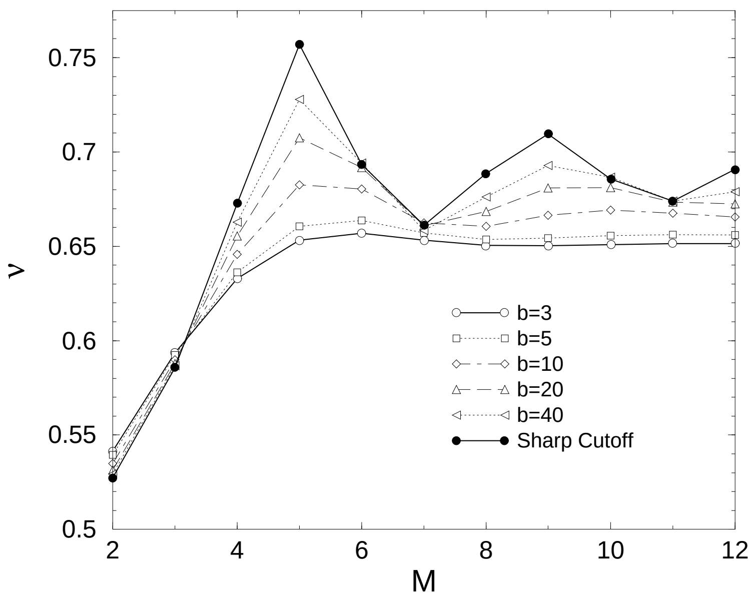

Figure 4. Critical exponent ν as a function of M using the exponential smearing function. Results for several values of b are shown for comparison.

C. Optimization

For a sharp cutoff, with a clear boundary between the high and the low modes, the IR cutoff is determined unambiguously as k. However, we wish to choose the cutoff to be as smooth as possible in order to minimize the generation of non-local, higher-derivative irrelevant interactions, as well as the order of truncation M. A general smooth smearing function $\rho_{k,\sigma}(p)$ results in a shift in the actual value of the IR cutoff, or the location of the peak of $-\partial\rho_{k,\sigma}(p)/\partial p$. When $\rho_{k,\sigma}(p)$ is too smooth, we no longer have a well-defined cutoff, i.e., the relation between the parameter k and the momenta of the modes appearing in the evolution equation.

As we have shown before, with the truncation scheme employed, the fixed-point solutions will vary with M and the would-be irrelevant operators will continue to contribute and evolve around the these approximate solutions. Thus, we would not expect *a priori* an accurate result for the critical exponents in this case. Nevertheless, we notice that for each of the three smearing functions discussed in Sec. II, there is an "optimal" value of σ for which ν converges most rapidly. For example, the optimal value for the exponential function is found to be $b = 3$, which incidentally coincides with the value of the extremum shown in Figure 2. When all three smearing functions are superimposed, it can be seen that they have approximately the same shape or smoothness, as illustrated in Fig. 5. That is, the most optimal smoothness does not depend on the detailed form of the smearing, i.e., hyperbolic tangent, exponential or power-law.

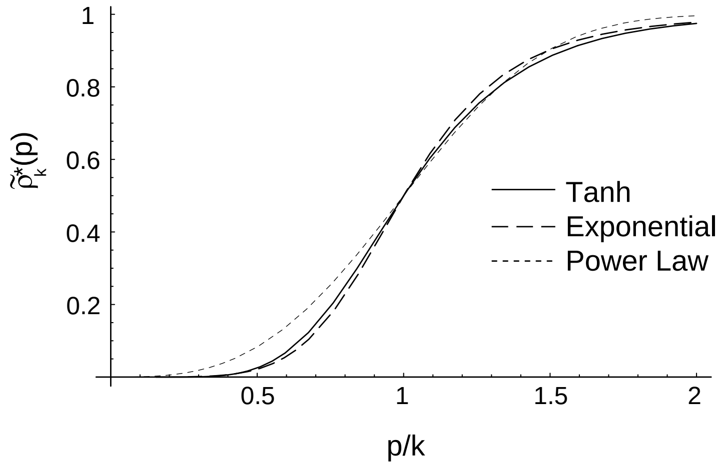

Figure 5. Comparison of the three optimized smearing functions.

The truncation dependence of ν for the three optimized smearing functions is ilustrated in Fig. 6. Although there remains small oscillations, we see a dramatic improvement in the convergence. In fact, the variation beyond $M = 7$ is only 1-2%. Our optimized value gives $\nu = 0.65(5)$ which is closer to the world's best value 0.631(2) [21] than 0.68(9) obtained by Aoki *et. al.* using an expansion around the moving minimum. We also remark that M, the number of operators involved in our calculation, is considerably smaller for a reasonably accurate estimate of ν [9]. However, by truncating the potential at $M = 12$, we have not been able to observe the oscillation of ν with M at a four-fold periodicity; an unambiguous observation of such behavior generally would require $M \geq 20$ [9][12].

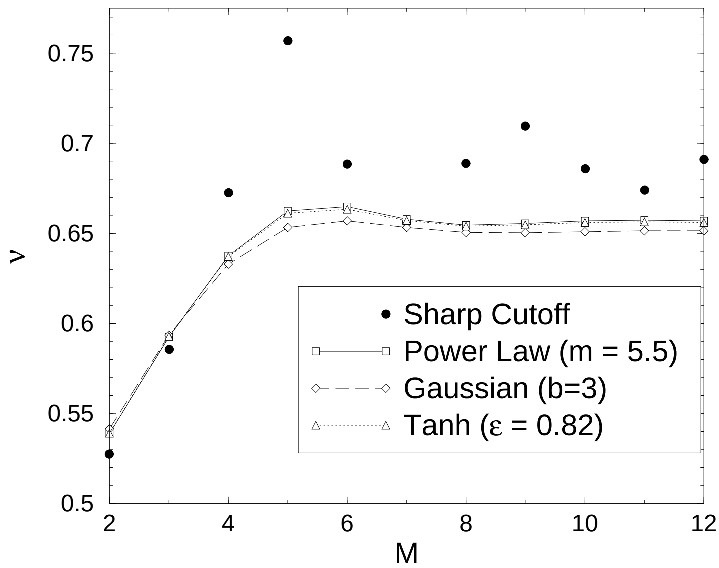

Figure 6. Critical exponent ν as a function of the level of polynomial truncation for all three optimized smearing functions and the sharp cutoff.

Why is σ related to the convergence of the series expansion? Physically, the parameter dictates the manner in which the quantum fluctuations modes are integrated over.

When σ is very small and the corresponding smearing function is too smooth, certain fast-fluctuating modes are only damped but not integrated over due to the lack of a well-defined boundary. In addition, some slowly-varying degrees of freedom which should be kept are integrated. As a result of the "mistreatment" of both the fast and the slow modes, the RG trajectory is greatly distorted, and the would-be irrelevant higher-order operators continue to evolve near the critical point obtained at order M.

The complicated interplay between the irrelevant operators can also be seen from the non-monotonic dependence of ν on σ observed in Figures 1 to 3. An extremum is found at σ_c for all orders of M. While for σ_c is a local maximum for $M = 2$, it becomes a local minimum for $M \geq 3$. The flip is clearly due to the inclusion of the irrelevant (marginal by power counting) $\bar{\Phi}^6$ operator.

In general, as σ increases and more fast modes are being included in the loop integrations, more cancellations between the effects of the irrelevant operators take place and the theory moves closer to the *true* Wilson-Fisher fixed point found in the exact approach. Nevertheless, when σ becomes too large and the resulting smearing function is too sharp, non-local effects begins to set in and the theory drifts away from the true RG trajectory. At $\sigma = \infty$ where the smearing function becomes a step function $\Theta(k-p)$, non-local effects become maximal and LPA is no longer adequate. One must then include derivative operators to all orders in the blocked action $\tilde{S}_k[\Phi]$ in order to arrive at a scheme-independent result.

Thus, we see that σ monitors the manner in which the irrelevant operators contribute to the flow of the theory. When the most optimal smoothness is reached, the maximum cancellation between these operators takes place, leading to the fastest convergence in the polynomial truncation.

In Sec. II we have also discussed an alternative mean approach, which is based on taking the sharp cutoff limit followed by a substitution of $\theta_0 = 1/2$ to avoid ambiguity, and yields Eq. (2.14) as the RG flow. We solve the equation and depict the result for ν in Figure 7.

Contrary to the sharp approach which results in an oscillatory behavior, ν is seen to converge rapidly. Such a change in the convergence is intimately related to the structure of the untruncated fixed-point potential $\bar{U}^*(\bar{\Phi})$ [5] which now obeys

$$-\frac{1}{2}\big(d-2\big)\bar{\Phi}\bar{U}^{*'}(\bar{\Phi}) + d\bar{U}^*(\bar{\Phi}) = -\frac{2\bar{U}^{*''}(\bar{\Phi})}{2+\bar{U}^{*''}(\bar{\Phi})}. \tag{4.1}$$

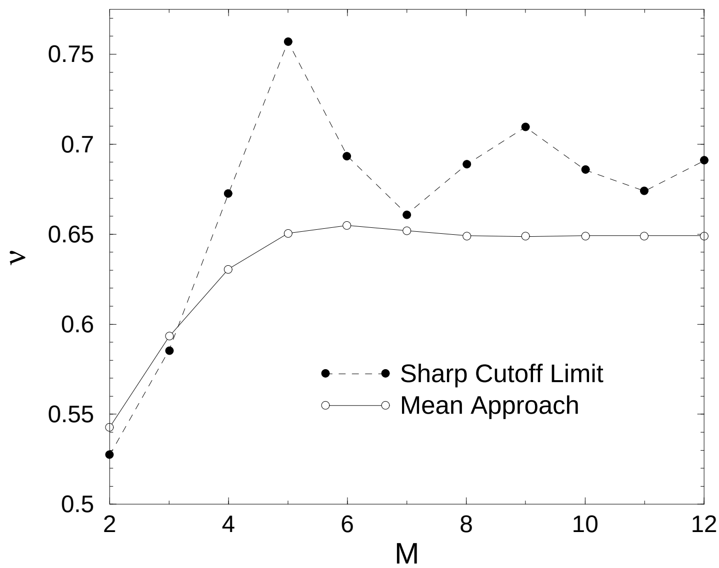

Figure 7. Critical exponent ν as a function of the level of polynomial truncation obtained with the mean approach and the sharp cutoff.

Even though Eq. (4.1) still cannot be solved analytically, its improvement in convergence can be shown by examining the right-hand-side of the equation representing the quantum corrections. By making a Taylor expansion of the expression with respect to $\bar{U}_k''(\bar{\Phi})$, we see that it has a radius of convergence $2/\bar{U}_k''$ which is twice that of $\ln[1+\bar{U}_k''(\bar{\Phi})]$ in Eq. (3.2). Since a general smooth cutoff always has a greater convergence radius, the resulting critical exponents will converge more rapidly compared with the sharp in the polynomial truncation scheme. To describe the amplitude and the periodicity of the oscillation more quantitatively, however, would require a detailed knowledge of the singularities of the untruncated potential $\bar{U}^*(\bar{\Phi})$.

V. SUMMARY AND DISCUSSIONS

In the present work we have demonstrated how one can improve the convergence of critical exponents calculated using polynomial truncations of the RG flow equation for the blocked potential $U_k(\Phi)$. Since in the sharp limit, ν oscillates with M, the order of truncation, we examined three parameterizations of a smooth cutoff function, hyperbolic tangent, exponential, and power-law, and made an attempt to eliminate as much as possible such unphysical artifacts. We find that there exists an optimal smoothness value which gives the most rapid convergence as a function of M. This is due to the maximal cancellation of the effects generated by the irrelevant operators in the blocked action. Our optimal smearing functions yield $\nu = 0.65(5)$ for $M \geq 7$ with a variation of 1-2% between the three cases. Had σ been too large, we would have to incorporate the higher-order derivative operators to account for the non-local effects generated in the course of RG evolution.

We have also learned from comparing the results for the sharp limit and the mean approach (depicted in Figure 7) that the convergence behavior of the polynomial truncation scheme is intimately related to the non-truncated solution of $\bar{U}_k(\bar{\Phi})$. The mean approach results in a larger radius of convergence, and hence a more rapid convergence. It remains to see if the same "trick" can be utilized for more complicated systems as well.

In light of the success of our optimized RG prescription, we can readily extend the formalism to address other issues such as the $O(N)$ models, the spinodal instability [22], gauge theories or chiral symmetry breaking. It would also be interesting to compare our usual momentum-cutoff approch with an alternative internal-space RG, which is a functional generalization of the Callan-Symanzyk equation, based on an infinitesimal variation of an internal-space parameter such as the mass scale [23].

ACKNOWLEDGMENTS

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